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NOTE

A NOTE ON PARTIAL PARALLEL CLASSES IN STEINER SYSTEMS

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A partial parallel class of blocks of a Steiner system $S(t, k, v)$ is a collection of pairwise disjoint blocks. The purpose of this note is to show that any $S(k, k+1, v)$ Steiner system, with $v \geqslant k^4+3k^3+k^2+1$, has a partial parallel class containing at least $(v-k+1)/(k+2)$ blocks.

1. Introduction

An $S(k, k+1, v)$ *Steiner system* is a pair (S, T) where S is a finite set of size v (called the *order* of the Steiner system) and T is a collection of $(k+1)$-element subsets of S (called *blocks*) such that every k-element subset of S belongs to *exactly one* block of T. An $S(2, 3, v)$ Steiner system is called a *triple system* and a $S(3, 4, v)$ Steiner system is called a *quadruple system*. It is well-known that a triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$ (see [4]) and in 1960 Hanani [2] proved that a quadruple system of order v exists if and only if $v \equiv 2$ or $4 \pmod 6$. Very little is known about $S(k, k+1, v)$ Steiner systems for $k \geqslant 4$. The reader is referred to the excellent bibliography on Steiner systems by Doyen and Rosa [1] for a fairly up-to-date account of what is known.

By a *partial parallel class* of blocks of the Steiner system (S, T) is meant a collection π of pairwise disjoint blocks of T. If the blocks of π partition S, then π is called a *parallel class* of blocks. The purpose of this note is to show that any $S(k, k+1, v)$ Steiner system, with $v \geqslant k^4+3k^3+k^2+1$, has a partial parallel class containing at least $(v-k+1)/(k+2)$ blocks.

2. A lower bound for partial parallel classes

In order to effect the proof of the above result we will need the following well-known facts concerning Steiner systems:

(1) If (S, T) is a $S(k, k+1, v)$ Steiner system then $|T| = \binom{v}{k}/(k+1)$.

(2) A partial $S(k, k+1, v)$ Steiner system is a pair (S, P) where P is a collection

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of $(k+1)$-element subsets of S such that every k-element subset of S belongs to *at most* one block of P. As a consequence, if (S, P) is a partial $S(k, k+1, v)$ Steiner system, then $|P| \leq \binom{v}{k}/(k+1)$.

(3) If (S, T) is a (partial) $S(k, k+1, v)$ Steiner system, then the *number* of blocks having *at least* one element in common with a given subset of $k+1$ elements is *at most* $(k+1)[\binom{v-1}{k-1}/k]$.

Theorem 2.1. *Let* (S, T) *be a* $S(k, k+1, v)$ *Steiner system, with* $v \geq k^4 + 3k^3 + k^2 + 1$. *Then* T *has a partial parallel class containing at least* $(v-k+1)/(k+2)$ *blocks.*

Proof. Let π be a partial parallel class of blocks of maximum size, say t, and denote by P the set of points belonging to the blocks of π. Since π is a partial parallel class of maximum size, every k-element subset of $S \setminus P$ belongs to a block of T which intersects P (in exactly one point). Denote by X the set of all such intersection points. For each $x \in X$ set $t(x) = \{b \setminus \{x\} : b \in T$ and $b \setminus \{x\} \subseteq S \setminus P\}$. Then each $(S \setminus P, t(x))$ is a partial $S(k-1, k, v-(k+1)t)$ Steiner system, $|t(x)| \leq \binom{v-(k+1)t}{k-1}/k$, and $\{t(x)\}_{x \in X}$ is a partition of the set of all k-element subsets of $S \setminus P$.
Claim: If b is a block of π containing at least 2 points of X, then for each $x \in X \cap b$ we must have $|t(x)| \leq k\binom{v-(k+1)t-1}{k-2}/(k-1)$. For if otherwise let y be any other point belonging to $X \cap b$ and b_1 and block in $t(y)$. Since at most $k\binom{v-(k+1)t-1}{k-2}/(k-1)$ of the blocks in $t(x)$ can intersect the block b_1, $t(x)$ must contain a block b_2 which is disjoint from b_1. But then $(\pi \setminus \{b\}) \cup \{b_1, b_2\}$ is a partial parallel class of blocks of larger size than π, a contradiction. It follows that for each block b of π containing at least 2 points of X, that $\sum_{x \in X \cap b} |t(x)| \leq (k+1)k\binom{v-(k+1)t-1}{k-2}/(k-1)$. Hence if we let A denote the number of blocks of π containing at least 2 points of X and B the number of blocks containing at most 1 point of X, then

$$\binom{v-(k+1)t}{k} = \sum_{x \in X} |t(x)|$$
$$\leq \left[(k+1)k\binom{v-(k+1)t-1}{k-2}\Big/(k-1)\right]A + \left[\binom{v-(k+1)t}{k-1}\Big/k\right]B.$$

There are two cases to consider:

(1) $$(k+1)k\binom{v-(k+1)t-1}{k-2}\Big/(k-1) \leq \binom{v-(k+1)t}{k-1}\Big/k.$$

Then

$$\binom{v-(k+1)t}{k} = \sum_{x \in X} |t(x)| \leq \left[\binom{v-(k+1)t}{k-1}\Big/k\right](A+B)$$
$$\leq \left[\binom{v-(k+1)t}{k-1}\Big/k\right]t$$

gives $t \geq (v-k+1)/(k+2)$.

(2) $$(k+1)k\binom{v-(k+1)t-1}{k-2}\Big/(k-1) > \binom{v-(k+1)t}{k-1}\Big/k.$$

In this case a simple calculation shows that $t \geq (v-k^3-k^2)/(k+1)$ which in turn is greater than or equal to $(v-k+1)/(k+2)$ for $v \geq k^4+3k^3+k^2+1$.

Combining cases (1) and (2) completes the proof of the theorem.

Theorem 2.1 says that every triple system of order $v \geq 45$ has a partial parallel class containing at least $\frac{1}{4}(v-1)$ blocks and that every quadruple system of order $v \geq 172$ has a partial parallel class containing at least $\frac{1}{5}(v-2)$ blocks. However, the following corollary shows that for triple systems we can replace $v \geq 45$ with $v \geq 9$, except possibly for $v = 15$, 19, and 27.

Corollary 2.2. *A Steiner triple system of order $v \geq 9$ has a partial parallel class containing at least $\frac{1}{4}(v-1)$ blocks (except possibly for $v = 15$, 19, and 27).*

Proof. For triple systems, case (2) of the theorem becomes $6 > \frac{1}{2}(v-3t)$. Hence $\binom{v-3t}{2} \leq 6A + [\frac{1}{2}(v-3t)]B \leq 6t$ which implies $t \geq \frac{1}{6}(2v+3-\sqrt{16v+9})$.

If we denote by $\langle x \rangle$ the smallest integer which is greater than or equal to x, then a case by case check shows that for $v = 9$, 13, 21, 25, 31, 33, 37, 39, or 43 $\langle\frac{1}{4}(v-1)\rangle = \langle\frac{1}{6}(2v+3-\sqrt{16v+9})\rangle$. Since t is an integer $\geq \frac{1}{6}(2v+3-\sqrt{16v+9})$ we must have $t \geq \frac{1}{4}(v-1)$.

3. Problems

Some of the following problems are due to A. Rosa.

(1) Show that the corollary holds for $v = 15$, 19, and 27; and obtain a similar result for quadruple systems.

(2) Improve the lower bound for partial parallel classes for both triple systems and quadruple systems.

(3) A *pairwise balanced design* (PBD) with block size k is a pair (P, B) where P is a finite set and B is a collection of k-element subsets of P (called *blocks*) such that every 2-element subset of P belongs to exactly one block of B. Find a good lower bound for the size of a partial parallel class in PBDs with block size $k \geq 4$ (see [3]).

(4) A PBD of index λ is a PBD in which every 2-element subset belongs to exactly λ blocks (no repeated blocks). For each $k \geq 3$ find the smallest λ such that a PBD of index λ and block size k has a partial parallel class of largest possible size.

(5) A Kirkman triple [illegible]

partitioned into parallel classes. It is well-known that a Kirkman triple system of order v exists if and only if $v \equiv 3 \pmod 6$ (see [5]). Find a lower bound on the size of partial parallel classes in Kirkman triple systems in which no two triples come from the same parallel class. Same problem for resolvable PBDs (see [3]).

References

[1] J. Doyen and A. Rosa, A bibliography and survey of Steiner systems, Boll. Un Mat. Ital. 7 (1973) 392–419.

[2] H. Hanani, On quadruple systems, Can. J. Math. 12 (1960) 145–157.

[3] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.

[4] T.P. Kirkman, On a problem in combinatorics, Cambridge and Dublin Math. J. 2 (1847) 191–204.

[5] D.K. Ray-Chaudhuri and R.M. Wilson, Solution of Kirkman's school girl problem, Am. Math Soc. Proc. Symp. Pure Math. 19 (1971) 187–204.